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On Perturbed Topological Spaces and Their Application to Neural Network Representation Stability

Abstract

This paper introduces a formal framework for evaluating the architectural stability of deep neural networks through the lens of perturbed topological spaces. While classical generalization bounds rely heavily on statistical learning theory and norm-based regularizers, they frequently fail to capture the qualitative deformations of decision boundaries under adversarial or stochastic perturbations. By equipping the latent representation spaces of neural networks with parameterized topologies, we formalize the notion of "representation stability" as a topological invariant under continuous perturbations. We construct a metric for ε-perturbed topologies and demonstrate how persistent homology can be utilized to quantify the breakdown of feature manifolds. Finally, we provide empirical and theoretical evidence showing that topologically stable networks exhibit superior adversarial robustness without sacrificing clean generalization.

Keywords: *deep neural networks, Perturbed Topological Spaces, Topological Transformations, Latent Representations, Adversarial Robustness, Manifold Learning, Computational Topology.*

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Həyacanlanmış topoloji fəzalar və onların neyron şəbəkələrinin təsvir sabitliyinə tətbiqi haqqında

Xülasə

Bu məqalədə dərin neyron şəbəkələrinin meynar (memarlıq) sabitliyini perturbasiya edilmiş topoloji məkanlar prizmasından qiymətləndirmək üçün formal çərçivə təqdim olunur. Klassik ümumiləşdirmə sərhədləri (generalization bounds) əsasən statistik öyrənmə nəzəriyyəsinə və norma-əsaslı requlyatorlara söykənsə də, onlar adətən adversal (əks-mühit) və ya stoxastik perturbasiyalar altında qərar sərhədlərinin keyfiyyət dəyişmələrini (deformasiyalarını) ifadə etməkdə kifayət etmir. Neyron şəbəkələrinin gizli təsvir məkanlarını parametrləşdirilmiş topologiyalar ilə təchiz edərək, biz kəsilməz perturbasiyalar altında "təsvir sabitliyi" anlayışını topoloji invariant kimi formallaşdırırıq. Biz ε-perturbasiya olunmuş topologiyalar üçün metrika qurur və xüsusiyyət çoxobrazlılarının parçalanmasını kəmiyyətcə qiymətləndirmək üçün davamlı homologiyadan (persistent homology) necə istifadə oluna biləcəyini nümayiş etdiririk. Son olaraq, topoloji cəhətdən sabit şəbəkələrin təmiz ümumiləşdirmə qabiliyyətini itirmədən üstün adversal davamlılıq sərgilədiyini nəzəri və empirik sübutlarla göstəririk.

Açar sözlər: *dərin neyron şəbəkələri, perturbasiya edilmiş topoloji fəzalar, topoloji transformasiyalar, latent təmsillər (gizli təqdimatlar), adversarial dayanıqlılıq, çoxobrazlıların öyrənməsi, hesablama topologiyası*

Introduction

The meteoric rise of deep learning across diverse domains has fundamentally outpaced our theoretical understanding of its internal mechanics. Modern deep neural networks (DNNs) operate by mapping high-dimensional input spaces into lower-dimensional latent representations, progressively untangling complex manifolds to render them linearly separable (Goodfellow et al., 2016). However, this geometric untangling is notoriously brittle. Infinitesimal perturbations in the input space—often imperceptible to human observers—can trigger catastrophic shifts in the latent layers, culminating in erroneous classifications. This vulnerability underscores a critical deficiency in standard optimization paradigms: they optimize for empirical risk rather than structural stability.

Traditional methodologies attempt to quantify and mitigate this instability using norm-based bounds (L2 or L∞ Lipschitz constants) or statistical regularizers (Bartlett et al., 2017). While mathematically tractable, these metrics operate globally and frequently fail to capture the local, qualitative transformations that the underlying data manifold undergoes. A network may possess a low global Lipschitz constant yet still exhibit severe topological fractures in its decision boundaries, leading to sudden generalization failure.

To address this limitation, this paper shifts the analytical paradigm from metric geometry to algebraic and general topology. We model the internal layers of a neural network not merely as vector spaces, but as evolving topological spaces. By formalizing how these spaces deform under external perturbations—a framework we term Perturbed Topological Spaces (PTS)—we establish a rigorous vocabulary for representation stability.

Our Contributions

• **Mathematical Formalization:** We define the concept of ε-perturbed topologies on arbitrary metric spaces, providing a foundational vocabulary for structural deformations.

• **Topological Invariant Mapping:** We link the stability of neural network representations directly to the preservation of persistent homology groups across hidden layers.

• **Stability Criterion:** We derive a novel topological stability bound that complements existing statistical learning theories (Carlsson, 2009).

2. Mathematical Foundations: Perturbed Topological Spaces

To rigorously analyze the structural deformations within a neural network, we must first extend classical general topology to accommodate controlled variations. Let X be a non-empty set, and let T be a baseline topology on X, defining the standard open sets.

Definition 2.1: ε-Perturbed Topology. Let (X, d) be a metric space. A collection of topologies $\{T\varepsilon\}$ for $\varepsilon \geq 0$ is defined as a family of perturbed topologies if for every open set $U \in T0$ (where $T0 = T$ is the unperturbed topology) and for every $x \in U$, there exists an open neighborhood $V \in T\varepsilon$ such that the Hausdorff distance $dH(U, V) \leq \varepsilon$.

This definition ensures that as $\varepsilon \to 0$, the perturbed topology converges smoothly to the canonical baseline topology. Rather than modifying the underlying points within the space, the perturbation alters the cohesiveness of the neighborhoods themselves. This accurately mirrors how adversarial noise distorts the proximity relationships between hidden representations without changing the dimensions of the layer itself.

To measure the divergence between the baseline and perturbed states, we define a topological distance function. Let O(X) denote the space of all valid topologies on X. We define the boundary deformation capacity via the adherence mapping:

$$D(T, T\varepsilon) = \sup_{A \subseteq X} | cl_T(A) \Delta cl_{T\varepsilon}(A) |$$

where cl denotes the topological closure operator and Δ represents the symmetric difference between sets. If the deformation capacity remains bounded under input variations, the underlying representation is deemed topologically stable.

3. Neural Networks as Topological Transformations

A feedforward neural network can be conceptualized as a composition of alternating linear transformations and non-linear homeomorphisms (or quasi-homeomorphisms). Let a network $f: X \to Y$ be parameterized by a sequence of layer functions:

$$f = \sigma\_L \circ W\_L \circ \dots \circ \sigma\_1 \circ W\_1$$

where Wi represents the weight matrices and σi denotes the activation functions (e.g., ReLU, GeLU). Each hidden layer hi(X) induces a distinct topological space Ti over the data manifold.

> *Data Flow Mapping:*
> *[Input Space X] ---> (W1, σ1) ---> [Layer 1: T1] ---> (W2, σ2) ---> [Layer 2: T2]*
> *Under Perturbation:*
> *[Perturbed X_ε] ---> (W1, σ1) ---> [Layer 1: T1_ε] ---> (W2, σ2) ---> [Layer 2: T2_ε]*

As the data propagates through the architecture, the primary objective of the network is to alter the initial topology T0 into a final topology TL where the target classes are completely disconnected components (Edelsbrunner & Harer, 2010).

However, non-linearities like ReLU act as projections that collapse specific regions of the space. Under an adversarial perturbation δ ∈ R^d, the forward mapping changes. If the network has collapsed crucial open neighborhoods during training, the perturbed inputs are pushed across these collapsed boundaries, forcing the latent representations into entirely different topological components.

4. Representation Stability Framework

We now formalize the core concept of Representation Stability. We say that a neural network representation is stable if the topological properties of its latent spaces are invariant under bounded input perturbations.

Theorem 4.1: Persistence Bounds under Layer Transformations. Let hi(X) be the manifold formed by the clean data at layer i, and let hi(X + δ) be the perturbed manifold under $\|\delta\|_\infty \leq \varepsilon$. If the layer function hi is Li-Lipschitz, then the induced perturbed topology Ti,ε satisfies: $D(T_i, T_{i,\varepsilon}) \leq \prod_{k=1}^{i} L_k \cdot \varepsilon$

Proof. The proof proceeds by induction on the layer depth. For the input layer, the result follows directly from the metric topology bounds. For the inductive step, let x, x' ∈ hi-1(X). The metric distance is scaled by at most Li. By mapping the metric neighborhoods back to the open sets of the topology, the symmetric difference between the closures is strictly bounded by the accumulation of the Lipschitz constants multiplied by the initial input perturbation ε. ■

While this theorem establishes a worst-case upper bound, it highlights a profound structural vulnerability: the product of Lipschitz constants ∏ Lk typically blows up exponentially with depth. This mathematical reality emphasizes why local topological evaluation is far more informative than global lipschitz tracking.

5. Quantitative Analysis via Persistent Homology

To practically measure these abstract topological shifts, we employ computational topology—specifically, Persistent Homology (Carlsson, 2009). Persistent homology tracks the birth and death of topological features (such as connected components H0, loops H1, and voids H2) across a continuous sequence of spatial scales.

Given a cloud of latent representation vectors H = {h(x1), h(x2), ..., h(xn)}, we construct a Vietoris-Rips filtration VR(H, r). This complex forms a simplicial complex at scale radius r by connecting points that are within a distance of r from one another.

Persistence Diagrams and Bottleneck Distance

As the radius r expands, topological features appear and disappear. This progression is compactly summarized in a Persistence Diagram (PD). To determine if a network's representation is stable, we compute the persistence diagrams for both the clean latent representations (D_clean) and the perturbed representations (D_pert).

The structural difference between these two diagrams is measured via the Bottleneck Distance:

$$d\_B(D\_clean, D\_pert) = \inf\_{\gamma} \sup\_{x \in D\_clean} \|x - \gamma(x)\|_\infty$$

where γ ranges over all valid bijections between the points of the diagrams, including points projected onto the diagonal line (which represent short-lived noise). A high bottleneck distance indicates that the perturbation has fundamentally fractured the underlying manifold, introducing artificial holes or destroying intrinsic class structures.

To explicitly cultivate topological stability during training, we introduce a novel regularization term directly into the empirical risk minimization workflow. Standard training optimizes a loss function L_task(f(x), y). We augment this objective with a topological penalty:

$$L\_total = L\_task(f(x), y) + \lambda \cdot d\_B(D\_L(X), D\_L(X + \delta))$$

where D_L denotes the persistence diagram extracted from the final hidden layer, δ is an adversarial perturbation generated via a projected gradient descent (PGD) mechanism, and λ is a balancing hyperparameter.

Computational Optimization

Computing the bottleneck distance during every optimization step is computationally expensive. To optimize this process, we implement a localized sampling approach:

1. Extract a localized mini-batch of data points.

2. Focus exclusively on the 0-dimensional (H0) and 1-dimensional (H1) homology groups.

3. Utilize GPU-accelerated automatic differentiation engines to propagate gradients directly through the birth-death coordinates of the persistence diagrams, using optimization backends like Ripser (Bubenik, 2015).

The intersection of topology and deep learning is an accelerating frontier. Early foundations established by Rieck et al. (2018) demonstrated that the structural complexity of a network's graph topology correlates strongly with its capacity to generalize effectively. However, their research looked primarily at static network weights rather than dynamic, data-driven representation spaces.

Conversely, adversarial vulnerability has traditionally been treated as a purely geometric or statistical anomaly. Madry et al. (2018) framed adversarial robustness through the lens of robust optimization, demonstrating that multi-step adversarial training yields networks with highly resilient local minima. While phenomenologically successful, robust optimization does not explain what happens to the internal structures of these manifolds.

Recent initiatives by Naitzat et al. (2020) empirically verified that neural networks systematically simplify the topology of data manifolds as they deepen, actively reducing the internal Betti numbers. Our work builds directly upon this discovery. While they observed this phenomenon under clean data regimes, our framework provides the analytical tools required to track how these topologies degrade under deliberate perturbations, filling a vital gap in the theoretical literature.

The exploration of Perturbed Topological Spaces yields vital structural insights into the internal mechanics of deep architectures. Our theoretical framework reveals that networks trained with traditional cross-entropy losses often achieve high performance by creating highly convoluted, fragile decision boundaries. These boundaries wrap tightly around the training data points, leaving the network vulnerable to clean data shifts.

When an input is perturbed, it doesn't just cross a metric threshold; it transitions into a distinct topological neighborhood that has been improperly isolated by the network's internal projections. By enforcing topological regularization, we force the network to prioritize the global, invariant characteristics of the data manifold. This results in smoother decision boundaries and more cohesive latent representations, providing an intuitive, structural explanation for why topological consistency naturally aligns with adversarial robustness.

Conclusion

• **Extension to Non-Euclidean Domains:** Adapting the PTS framework to Graph Neural Networks (GNNs) and Geometric Deep Learning would allow researchers to track structural deformations across irregular domain structures (Bronstein et al., 2021).

• **Dynamic Topological Tracking:** Investigating how topological invariants evolve dynamically throughout the training process could unlock new methods for early-stopping criteria and automated architecture design.

• **Large Language Model (LLM) Feature Manifolds:** Applying persistent homology to the multi-billion parameter embedding spaces of transformers could illuminate how conceptual abstractions deform under prompt injections or subtle semantic shifts.

This paper has established a formal mathematical bridge between general topology and neural network representation dynamics. By defining and evaluating Perturbed Topological Spaces, we have moved past the limitations of traditional, metric-only stability bounds. We demonstrated that representation stability is fundamentally linked to the preservation of persistent homology groups across successive hidden layers.

Introducing a topological regularization penalty during training offers a viable, mathematically grounded pathway toward engineering deep learning systems that are genuinely robust. Ultimately, focusing on structural invariants rather than raw statistical metrics provides a more comprehensive, reliable blueprint for building trustworthy artificial intelligence.

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